University of Heidelberg

Department of Economics

Discussion Paper Series | No. 453

Factor Substitution, Income Distribution, and Growth in a Generalized Neoclassical Model

Andreas Irmen and
Rainer Klump

October 2007

FACTOR SUBSTITUTION, INCOME DISTRIBUTION, AND GROWTH IN A GENERALIZED NEOCLASSICAL MODEL

Andreas Irmen*

University of Heidelberg, CEPR, London, and CESifo, Munich

Rainer Klump†

University of Frankfurt, and CFS, Frankfurt

Abstract: We analyze a generalized neoclassical growth model that combines a normalized CES production function and possible asymmetries of savings out of factor incomes. This generalized model helps to shed new light on a recent debate concerning the impact of factor substitution and income distribution on economic growth. We can show that this impact relies on both an efficiency and an acceleration effect, where the latter is caused by the distributional consequences of an increase in the elasticity of substitution. While the efficiency effect is always positive, the direction of the acceleration effect depends on the particular savings hypothesis. However, if savings out of capital income are substantial so that a certain threshold value is surpassed we find that the efficiency effect dominates so that higher factor substitution can work as a major engine of growth.

Keywords: Capital Accumulation, Elasticity of Substitution, Income Distribution, Neoclassical Growth Model.

JEL-Classification: E21, O11, O41.

This Version: October 16, 2007.

*University of Heidelberg, Alfred-Weber-Institute, Grabengasse 14, D-69117 Heidelberg, Germany, airmen@uni-hd.de.

†University of Frankfurt, Department of Economics, Schumannstraße 60, D-60325 Frankfurt on the Main, Germany, klump@wiwi.uni-frankfurt.de.

1 Introduction

Aggregate models of economic growth are based on two pillars, namely an assumption how to model aggregate production and an assumption how capital accumulation is fueled by aggregate saving decisions. The first pillar centers around the concept of substitutability between factor inputs, whereas the second deals with the distribution of factor incomes. Some prominent examples show that the assumptions on both of these pillars can be either very general or rather narrow. The standard neoclassical growth model by Solow (1956) combines the very general concept of an aggregate production function that allows for substitutability between the factors labor and capital with a constant saving ratio out of total factor income. As a general functional form to model aggregate production with various degrees of factor substitution, Solow invented what later became known as the CES production function (Arrow, Chenery, Minhas, and Solow (1961)). In contrast, the growth model proposed by Kaldor (1956) is based on the narrow Post-Keynesian assumption of a limitational production function, but at the same time assumes quite generally that savings out of labor and capital income differ. As a consequence, aggregate saving is strongly influenced by the distribution of factor incomes. The latter is also central to the overlapping-generations (OLG) version of the neoclassical growth model (Diamond (1965)), where aggregate saving stems solely from wage income. Similarly, it matters under the "classical savings hypothesis," as used, e. g., by Uzawa (1961), that regards only capital income as the source of aggregate savings.

In this paper, we explore a neoclassical growth model that incorporates the most general assumptions on both pillars, i. e., various degrees of substitutability in the aggregate production function and possible asymmetries of savings out of factor incomes. Our generalized model helps to shed new light on a recent debate concerning the impact of factor substitution and income distribution on economic growth. This debate began with the contributions by de La Grandville (1989) and Klump and de La Grandville (2000) studying the link between the elasticity of substitution, being treated as a parameter of a normalized CES production function, and growth in the Solow model. They come to the conclusion that the degree of factor substitution is a powerful engine of growth in the sense that a higher elasticity of substitution leads to a higher growth rate and a higher steady-state level of per-capita income. The relevance of this conclusion has been challenged when Miyagiwa and Papageorgiou (2003) explored the growth effects of the elasticity of substitution in a discrete-time OLG framework and did not find a monotonic relationship but rather report cases where a higher elasticity of substitution would also have a negative impact on growth. These results have been confirmed by Irmen (2003) in the context of a Diamond-type growth model set out in continuous time. His explanation of the controversial results is based on the distinction between two effects which are caused by a change in the aggregate elasticity of factor substitution: an *efficiency effect* by which changes in factor substitution influence the productivity of factor inputs and an *acceleration effect* that relates changes in factor substitution to the evolution of the capital intensity.

In our general model, we demonstrate more broadly how these two effects interact. Moreover, we show that a *distribution effect* surfaces in the presence of differing savings rates out of wage and profit/capital incomes. As a consequence the direction of the acceleration effect depends on the particular saving hypothesis. If the distribution effect is negative, the sign of the overall effect results from its strength relative to the efficiency effect. In growing Diamond-type economies, the tension between these two countervailing forces explains the negative acceleration effect reported in Irmen (2003) and Miyagiwa and Papageorgiou (2003). In the general case, where also capital income is a source of aggregate savings, the direction and size of the overall effect is determined not only by the elasticity of substitution and the different saving ratios but also by the baseline values for capital, production per capita, and the income distribution. Moreover, as long as the savings ratio out of profit income is not lower than the savings ratio out of wages, or that it at least surpasses a certain threshold value conditional on the various parameters of the model, the growth effects of higher factor substitution remain positive.

The rest of the paper is organized as follows. Section 2 briefly highlights and recalls some important analytical properties of normalized CES production functions. In particular, we clarify in what sense the normalized CES allows us to isolate the effect of the elasticity of substitution on the growth process. Section 3 introduces our generalized neoclassical growth model and studies the effects of changing the elasticity of substitution on the growth process. We start by looking at local effects and then proceed to a global analysis. Section 4 concludes. All proofs are relegated to an appendix.

2 Normalized CES Production Function and Per-Capita Output

We consider the following CES per-capita production function

$$y = f(k) = A\left[ak^{\psi} + (1-a)\right]^{1/\psi}, \tag{1}$$

with $A > 0$, $1 > a > 0$, $1 > \psi > -\infty$, and k denoting the capital-labor ratio. The parameter $\sigma = 1/(1-\psi)$ is the elasticity of substitution. Following Klump and de La Grandville (2000), we normalize (1) by choosing some baseline capital-labor ratio $\overline{k} > 0$, some level of per-capita output $\overline{y}$, and a marginal rate of substitution $\overline{m} > 0$. The normalized CES production function that satisfies these criteria can then be computed to equal

$$y = f_\sigma (k) = A(\sigma) \left[a(\sigma) k^\psi + (1 - a(\sigma)) \right]^{1/\psi} \tag{2}$$

with

$$A(\sigma) \equiv \overline{y} \left(\frac{\overline{k}^{1-\psi} + \overline{m}}{\overline{k} + \overline{m}} \right)^{1/\psi} \qquad \text{and} \qquad a(\sigma) \equiv \frac{\overline{k}^{1-\psi}}{\overline{k}^{1-\psi} + \overline{m}}. \tag{3}$$

The normalization implies a capital share $\pi_\sigma(k)$ with a baseline value $\pi_\sigma\left(\bar{k}\right) \equiv \bar{\pi}$, such that (2) may also be written as

$$y = f_\sigma (k) = \frac{\bar{y}}{\bar{k}} \left(\frac{\bar{\pi}}{\pi_\sigma} \right)^{1/\psi} k, \tag{4}$$

where

$$\bar{\pi} \equiv \frac{\bar{k}}{\bar{k} + \bar{m}} \quad \text{and} \quad \pi_\sigma(k) \equiv \frac{k^\psi \bar{k}^{1-\psi}}{k^\psi \bar{k}^{1-\psi} + \bar{m}}. \tag{5}$$

This alternative representation of the normalized CES function emphasizes that the initial functional income distribution and its evolution play a central role for the evolution of the economy.

In what follows, we denote partial derivatives with respect to k by a prime so that $f'_\sigma := \partial f_\sigma / \partial k$ and $\partial f''_\sigma / \partial \sigma := \partial^3 f_\sigma / \partial^2 k \partial \sigma$. If not indicated otherwise, the argument of f_σ is k.

The interpretation that we can attach to changes of σ is based on the following implication of the above normalization.

Lemma 1 The normalized CES production function $f_\sigma(k)$ as given by (2) satisfies

$$\frac{\partial f''_\sigma\left(\bar{k}\right)}{\partial \sigma} > 0. \tag{6}$$

Lemma 1 provides the first key to the understanding of the growth effects of the elasticity of substitution: at $\overline{k}$ there is an inverse relationship between the elasticity of substitution and the curvature of the normalized CES production function. This relationship has an interpretation in terms of the degree of complementarity of both input factors. Let $Y_\sigma = F_\sigma(K, L)$ be the CES production function underlying (2) with K and L denoting aggregate capital and labor inputs. Then,

$$F_{\sigma KL} \equiv \frac{\partial^2 F_\sigma(K, L)}{\partial K \partial L} = -k f''_\sigma(k)$$

for all admissible (K, L) and $k \equiv K/L$. Moreover,

$$\left.\frac{\partial F_{\sigma KL}}{\partial \sigma}\right|_{(K,L)=\left(\overline{K},\overline{L}\right)} = -\overline{k}\,\frac{\partial f''_\sigma\left(\bar{k}\right)}{\partial \sigma} < 0. \tag{7}$$

Therefore, at $\overline{k}$ a higher elasticity of substitution implies a *lower degree of complementarity between capital and labor*. We show below that this property of the normalized CES, in conjunction with the savings hypothesis, drives the dynamics of capital accumulation and per-capita income in the neighborhood of the baseline value $\bar{k}$.

Klump and de La Grandville (2000), Theorem 1, establishes a key global property of (2), namely,

$$\frac{\partial f_\sigma(k)}{\partial \sigma} > 0 \text{ for } k \neq \overline{k}. \tag{8}$$

Hence, the elasticity of substitution has an interpretation as "a measure of the efficiency of the productive system" (de La Grandville (1989), p. 479) in the sense that the higher σ, the higher is per-capita output for any capital-labor ratio other than $\overline{k}$. We shall refer to (8) as the *efficiency effect*.

3 The Generalized Neoclassical Growth Model

We consider a competitive economy in continuous time, i. e., $t \in [0, \infty)$. If not indicated otherwise the baseline values $\bar{k}, \bar{y}$, and $\bar{m}$ can be viewed as initial values of the economies under scrutiny. Marginal cost pricing implies a real wage and a real rate of return on capital equal to

$$w_\sigma(k) = f_\sigma(k) - k f'_\sigma(k), \tag{9}$$

$$r_\sigma(k) = f'_\sigma(k). \tag{10}$$

3.1 Factor Substitution, Income Distribution, and Capital Accumulation

In our generalized model of growth which combines factor substitution à la Solow with possible asymmetries in savings out of factor incomes à la Kaldor, the speed of capital accumulation is determined by the following equation of motion

$$\begin{aligned} \dot{k} &= s^w \left[f_\sigma(k) - k f'_\sigma(k) \right] + s^r \left[k f'_\sigma(k) \right] - nk, \\ &= \left[s^w (1-\pi_\sigma) + s^r \pi_\sigma \right] f_\sigma(k) - nk, \\ &= \left[s^w (1-\pi_\sigma) + s^r \pi_\sigma \right] \frac{\bar{y}}{\bar{k}} \left(\frac{\bar{\pi}}{\pi_\sigma} \right)^{1/\psi} k - nk. \end{aligned} \tag{11}$$

Here, $s^w, s^r \in (0,1)$ denote the marginal and average savings rates out of wage and capital/profit income, respectively, and n is the growth rate of the employed labor force. Moreover, equations (9), (10), and (4) were used to derive the expressions.

From (11), it is straightforward to derive the *acceleration effect* of the elasticity of substitution, i. e., its influence on the speed of capital accumulation, as

$$\frac{\partial \dot{k}}{\partial \sigma} = \left[s^w (1-\pi_\sigma) + s^r \pi_\sigma \right] \frac{\partial f_\sigma(k)}{\partial \sigma} + f_\sigma(k) \left(s^r - s^w \right) \frac{\partial \pi_\sigma}{\partial \sigma}. \tag{12}$$

Equation (12) allows for a basic insight into the mechanics of our generalized neoclassical growth model. According to (8), the first term on the right-hand side is always positive and reflects the *efficiency effect* of a higher degree of factor substitution. A higher elasticity of substitution increases per-capita income which, for a given income distribution, raises savings. The second term on the right-hand side of (12) reflects the *distribution effect*, $\partial \pi_\sigma / \partial \sigma$. For a given level of per-capita output, this term captures the impact of the elasticity of substitution on the functional income distribution, and, in turn, on aggregate savings. We know from Klump and de La Grandville (2000) (see, their equation 10) that

$$\frac{\partial \pi_\sigma}{\partial \sigma} = \frac{\pi_\sigma (1-\pi_\sigma)}{\sigma^2} \ln \left(\frac{k}{\bar{k}} \right). \tag{13}$$

Klump and Saam (2008) propose that the baseline capital intensity corresponds to the capital intensity that would be efficient if the economy's elasticity of substitution were zero. For $k > \bar{k}$ the economy's relative bottleneck resides in its capacity to make productive use of additional capital. Relaxing this bottleneck by allowing for higher factor substitution (or lower complementarity) would then increase the capital income share. For $k < \bar{k}$ the same would be true for labor and its income share.

Hence, for $k > \bar{k}$ a rise in the elasticity of substitution raises the capital share. If, in addition, $s^r > s^w$, then such a rise shifts the income distribution in favor of capital income out of which a larger fraction is saved. Then, the channel via the efficiency effect and the one via the distribution effect are positive.[1] As a result, aggregate saving increases and $\partial\dot{k}/\partial\sigma > 0$, i. e., the *acceleration effect* of the elasticity of substitution is positive.

3.2 Local Effects of Higher Factor Substitution

Clearly, the acceleration effect need neither be positive nor monotonic for all $k > 0$ and $\sigma > 0$. However, the following proposition establishes that in a small neighborhood of $\bar{k}$, the acceleration effect is indeed monotonic for all admissible values of k. Moreover, its driving force is the change in the degree of complementarity identified in Lemma 1.

Proposition 1 Let k belong to a sufficiently small neighborhood of $\bar{k}$ and define $\dot{k} = s^w\, w_\sigma(k) + s^r\, k\, r_\sigma(k) - nk \equiv \dot{k}_\sigma(k)$.

1. If $s^w \neq s^r$, then

$$\frac{\partial \dot{k}_\sigma(k)}{\partial\sigma} \simeq \frac{\partial f''_\sigma\left(\bar{k}\right)}{\partial\sigma}\,\bar{k}\,(s^r - s^w)\,(k-\bar{k}). \tag{14}$$

2. If $s^w = s^r = s$, then

$$\frac{\partial \dot{k}_\sigma(k)}{\partial\sigma} \simeq \frac{s}{2}\,\frac{\partial f''_\sigma\left(\bar{k}\right)}{\partial\sigma}\,(k-\bar{k})^2 > 0. \tag{15}$$

According to Proposition 1, the acceleration effect is monotonic in the neighborhood of $\bar{k}$, i. e., either positive or negative for all admissible k. In the general case, if $s^w \neq s^r$, what matters is how the savings rates relate to the change in the relative scarcity of capital. More precisely, we have

$$\frac{\partial \dot{k}_\sigma(k)}{\partial\sigma} \gtrless 0 \quad \Leftrightarrow \quad (s^r - s^w)\,(k-\bar{k}) \gtrless 0. \tag{16}$$

This result generalizes previous findings derived for $s^r = 0$ to the case where $s^r > 0$ (see, e. g., Irmen (2003), Proposition 1). The presence of the term $s^r - s^w$ suggests that the distribution effect drives the sign of the acceleration effect. Indeed, for a growing economy where $k > \bar{k}$, we learn from the proof of Proposition 1 that a

[1] For the same reasons, the channel via the distribution effect is also positive if $k < \bar{k}$ and $s^r < s^w$.

rise in σ increases the rate of return on capital and lowers the wage. Since there is no first-order effect of σ on aggregate income at $\bar{k}$, the wage income falls whereas capital income increases. As a consequence, the acceleration effect is positive for $s^r > s^w$ and negative for $s^w > s^r$.

In the usual neoclassical (Solow) case, where $s^w = s^r = s$, the distribution effect has no bite. For this case, Proposition 1 provides a new (local) rationale for Theorem 1 in Klump and de La Grandville (2000): the comparative static of Lemma 1 has initially a positive second-order effect on the speed of capital accumulation. Therefore, the sign of the acceleration effect is positive for growing and shrinking economies, i. e., it is independent of $k \gtrless \bar{k}$.

We can use these findings to determine the local effect of the elasticity of substitution on the evolution of per-capita income. From $y(t) = f_\sigma(k(t))$, we have $\dot{y}(t) = f'_\sigma(k(t))\,\dot{k}(t)$ such that

$$\frac{\partial \dot{y}}{\partial \sigma} = \frac{\partial f'_\sigma}{\partial \sigma}\,\dot{k} + f'_\sigma\,\frac{\partial \dot{k}}{\partial \sigma}. \tag{17}$$

The right-hand side shows two channels. First, each unit of capital accumulated between today and tomorrow may have a higher or a lower marginal product depending on whether the *marginal productivity effect* $\partial f'_\sigma / \partial \sigma$ is positive or not. Second, for a given marginal product of capital, the amount of capital accumulated between today and tomorrow changes in accordance with the *acceleration effect*.

Proposition 2 Let k belong to a sufficiently small neighborhood of $\bar{k}$ and define $\dot{y} = f'_\sigma(k)\,\dot{k}_\sigma(k) \equiv \dot{y}_\sigma(k)$. Then,

$$\frac{\partial \dot{y}_\sigma(k)}{\partial \sigma} \gtrless 0 \quad \Leftrightarrow \quad \left(2\bar{\pi}\left(\frac{s^r}{s^w} - 1\right) + \frac{1}{s^w \bar{y}}\left(s^w \bar{y} - n\bar{k}\right)\right)\left(k - \bar{k}\right) \gtrless 0 \tag{18}$$

Proposition 2 encompasses several interesting cases. For instance, in a growing economy, where $s^r > s^w$, the expression (18) is strictly positive. Hence, economies with a higher elasticity of substitution have a higher per-capita income as long as k remains in the admissible neighborhood. This finding is quite intuitive since in this scenario the acceleration effect is positive by Proposition 1, and a higher factor substitution increases the marginal product of capital (see, equation (34) in the proof of Proposition 2).

Moreover, Proposition 2 may be used to determine a critical savings rate, s^r_c, such that $\partial \dot{y}_\sigma(k) / \partial \sigma > 0$ in a growing economy. One finds that

$$s^r > s^r_c \equiv \max\left[s^w\left(1 - \frac{1}{\bar{\pi}}\right) + \frac{n\bar{k}}{\bar{\pi}\bar{y}},\, s^w\left(1 - \frac{1}{2\bar{\pi}}\right) + \frac{n\bar{k}}{2\bar{\pi}\bar{y}},\, 0\right] \tag{19}$$

is sufficient for this. Here, the first term in brackets makes sure that $\dot{k}\left(\bar{k}\right) > 0$, and the second assures that the effect of (18) is positive for $k > \bar{k}$. The critical savings rate depends, inter alia, on the chosen baseline values and may fall short of s^w. If we conclude, invoking the empirical findings of e. g. Bernanke and Gürkaynak (2001), that $\bar{\pi} \approx 1/3$, and take $n \approx 0$ as an approximation for many industrialized countries, condition (19) is satisfied whenever $s^r > 0$.

If $s^r = 0$, Proposition 2 predicts that economies with a higher elasticity of substitution may have a lower per-capita income. For instance, in a growing economy with $\bar{\pi} < 1/2$, the precise condition for $\partial \dot{y}_\sigma\left(k\right)/\partial\sigma < 0$ is $n\bar{k}/\bar{y}(1-2\bar{\pi})^{(-1)} > s^w > n\bar{k}/\bar{y}$. The second inequality assures that the economy initially grows. The first makes sure that the effect in (18) is strictly negative for $k > \bar{k}$. Intuitively, in a growing economy this possibility arises since the acceleration effect of Proposition 1 becomes negative for $k > \bar{k}$. This finding confirms results found by Irmen (2003) and Miyagiwa and Papageorgiou (2003) for Diamond-like economies.

For a Solow economy, Proposition 2 is consistent with the findings of Klump and de La Grandville (2000). Indeed, for $s^r = s^w = s$, (18) reduces to

$$\frac{\partial \dot{y}_\sigma\left(k\right)}{\partial\sigma} > 0 \quad \Leftrightarrow \quad \dot{k}_\sigma\left(\bar{k}\right)\left(k-\bar{k}\right) > 0, \tag{20}$$

i. e., a higher elasticity of substitution means a higher per-capita income independent of whether the economy grows or shrinks.

The local analysis of this section supports the conclusion that the impact of a higher elasticity of substitution on the evolution of per-capita income is positive even if the saving rate s^r and s^w differ. While a negative acceleration effect can occur in Diamond-like economies and a negative total effect can therefore not be excluded theoretically, it seems that the empirically relevant case is the one where savings out of capital income are so important that the savings rate out of capital income exceeds the critical threshold value. Moreover in a growing economy, it is sufficient for a positive total effect that the savings rate out of profit income is not lower than the savings rate out of wage income.

It is worth noting that the local analysis of Propositions 1 and 2 may capture the properties of an economy's global dynamics. For instance, this is the case if the economy converges to a steady-state, k^*, that is part of the admissible neighborhood of $\bar{k}$. Much of the trust that growth economist have when they study the local dynamics of a steady state rests on this assumption. Of course, the analysis also applies to the extreme case where $\bar{k} = k^*$. However, then by definition the steady state can no longer depend on the elasticity of substitution.

3.3 Global Effects

We are now able to proceed to an explicit analysis of global effects of a higher elasticity of substitution on growth given possible asymmetries in the saving ratios. Our results can be regarded as generalizations of the two basic theorems that appear in Klump and de La Grandville (2000).

Proposition 3 Consider two economies that initially differ only with respect to their elasticity of substitution. Moreover, assume that $\dot{k}_{\sigma_1}\left(\bar{k}\right) = \dot{k}_{\sigma_2}\left(\bar{k}\right) > 0$, where $\sigma_2 > \sigma_1$.

If $s^r \geq s^w$, then the economy with the higher elasticity of substitution has a larger capital stock and a higher per-capita income for all $t > 0$.

Again we see here how the interplay between the efficiency effect, the distribution effect and capital accumulation works. A higher elasticity of substitution leads to a higher efficiency of total factor inputs and also (for $k > \bar{k}$) to an increase in the profit share. If savings stemmed from wage incomes only as it is the case in Diamond-like economies, this redistribution would slow down capital accumulation and could, in the worst case, make the capital intensity decline. According to (12), $s^r \geq s^w$ is sufficient for a positive acceleration effect.

For a clear-cut global result concerning the evolution of per-capita income in a growing economy, we need more than a positive acceleration effect. In accordance with (17), what matters in addition is how the marginal product of capital responds to a rise in the elasticity of substitution. The proof of Proposition 3 establishes that this effect is indeed strictly positive, i. e., $\partial f'_\sigma/\partial\sigma > 0$ for all $k > \bar{k}$. Hence the intuition associated with the efficiency effect of (8) extends to the marginal product of capital when $k > \bar{k}$.

In the Solow economy underlying Theorem 1 of Klump and de La Grandville (2000), the redistribution of incomes has no effect on total savings. An important implication of our Proposition 3 is that the qualitative results of this theorem survive in an environment with differing saving rates as long as empirically plausibel values are employed, i. e., if $s^r > s^w$.

Next, we turn to the analysis of the effect of the elasticity of substitution on the steady-state per-capita income. Let k^* denote a steady state capital intensity and $\pi^*_\sigma \equiv \pi^*_\sigma(k^*) \in (0, 1)$ the corresponding capital share. From (11), a steady state must satisfy

$$\dot{k} = 0 \quad \Leftrightarrow \quad \frac{f_\sigma(k^*)}{k^*}\left[s^w\left(1 - \pi^*_\sigma\right) + s^r \pi^*_\sigma\right] = n. \tag{21}$$

To study the effect of the elasticity of substitution on k^*, we totally differentiate (21). This gives

$$\frac{dk^*}{d\sigma} = \frac{\frac{s^w}{k^*}\frac{\partial f_\sigma(k^*)}{\partial \sigma} + \left(s^r - s^w\right)\frac{\partial f'_\sigma(k^*)}{\partial \sigma}}{\frac{s^w f_\sigma(k^*)}{(k^*)^2}(1-\pi^*_\sigma) - \left(s^r - s^w\right) f''_\sigma(k^*)} \tag{22}$$

and leads to the following results.

Proposition 4 Consider two economies that initially differ only with respect to their elasticity of substitution. Moreover, assume that a steady state for both economies exists and that $\dot{k}_{\sigma_1}\left(\bar{k}\right) = \dot{k}_{\sigma_2}\left(\bar{k}\right) > 0$, where $\sigma_2 > \sigma_1$.

If $s^r \geq s^w$, then the economy with the higher elasticity of substitution has a larger steady-state capital stock and a higher steady-state per-capita income.

As long as the savings ratio out of capital income is large enough to overcome possible negative distributional effects of a higher elasticity of substitution on aggregate savings, higher factor substitution induces higher steady-state values of the economy. Again, it is sufficient that both savings ratios are equal as in the Solow model. This is the point of Theorem 2 in Klump and de La Grandville (2000). Our Proposition 4 shows that the qualitative results of this theorem extend to economies where $s^r \geq s^w$ and $k^* > \bar{k}$.

4 Concluding Remarks

Since all models of economic growth combine assumptions about the substitutability between factors of production with a hypothesis about savings from factor incomes, the interaction between factor substitution and capital accumulation is the basic engine of growth. The standard neoclassical growth model, working typically with a Cobb-Douglas production function (and thus an elasticity of substitution equal to one) and a constant savings ratio of total factor income, does not allow for an in-depth analysis of this interaction. We therefore propose a generalized neoclassical growth model, in which a normalized CES production function identifies the effect of a variation in the elasticity of substitution between capital and labor, and where the savings hypothesis explicitly includes the possibility of asymmetries in savings out of capital and labor incomes. This general framework then encompasses neoclassical, classical, Post-Keynesian, and OLG-like settings as special cases.

Our results show that the impact of a higher degree of factor substitution on capital accumulation and growth depends on two separate effects. While the efficiency effect is always positive and independent of any savings hypothesis, the accumulation effect can be positive or negative depending on the distributional consequences of higher factor substitution and on the assumed sources of savings. In the special case of a growing Diamond economy, where all savings come out of labor income, a higher elasticity of substitution squeezes the total rate of capital accumulation by reducing the labor share in total income. If this effect dominates the increase in total income resulting from the efficiency effect, then the overall effect on growth would be negative. We are able to show, however, that this constellation is rather unlikely to occur. As long as the savings ratio out of profits is not lower than the savings ratio out of wages or that it at least surpasses a certain lower threshold value, the growth effects of higher factor substitution remain positive as pointed out by Klump and de La Grandville (2000).

Miyagiwa and Papageorgiou (2003), p. 161, concluded from their analysis of the OLG-model that "… whether the elasticity of substitution has a positive or negative effect on economic growth depends on our view of the world, that is, on the particular framework (Solow vs. Diamond) we believe to be a better representation of the world." Our analysis leads now to a more precise conclusion. As long as in the real world we find significant savings out of capital income the interaction between factor substitution, capital accumulation and growth is much better approximated by the Solow framework than by the Diamond setting. Moreover, our generalized growth model can help to reveal the complex mechanics that make the elasticity of substitution a powerful engine of growth.

5 Appendix

5.1 Proof of Lemma 1

Due to the normalization we have at $\overline{k}$:

$$\overline{y} = f_\sigma\left(\overline{k}\right) \text{ and } \overline{m} = m_\sigma\left(\overline{k}\right) = \left(f_\sigma\left(\overline{k}\right) - \overline{k} f_\sigma'\left(\overline{k}\right)\right) / f_\sigma'\left(\overline{k}\right)$$

so that the slope of f_σ at $\overline{k}$ is

$$f_\sigma'\left(\overline{k}\right) = \frac{f_\sigma\left(\overline{k}\right)}{\overline{k} + \overline{m}}. \tag{23}$$

The elasticity of substitution is defined as

$$\sigma \equiv -\frac{f_\sigma'\left(f_\sigma - k f_\sigma'\right)}{k f_\sigma f_\sigma''}. \tag{24}$$

Hence,

$$f_\sigma'' = -\frac{f_\sigma'\left(f_\sigma - k f_\sigma'\right)}{k f_\sigma \sigma}. \tag{25}$$

and

$$\frac{\partial f_\sigma''\left(\bar{k}\right)}{\partial \sigma} = \frac{f_\sigma'\left(f_\sigma - k f_\sigma'\right)}{k f_\sigma \sigma^2} = \frac{-f_\sigma''\left(\overline{k}\right)}{\sigma} > 0. \tag{26}$$

From the concavity of f_σ we have $f_\sigma''\left(\overline{k}\right) < 0$ which proves (6).

QED.

5.2 Proof of Proposition 1

The components of $\dot{k}_\sigma(k)$ stem from equations (9), (10), and (11). Then, the acceleration effect of the elasticity of substitution, is

$$\frac{\partial \dot{k}}{\partial \sigma} = s^w \frac{\partial w_\sigma\left(k\right)}{\partial \sigma} + s^r k \frac{\partial r_\sigma\left(k\right)}{\partial \sigma} \equiv \frac{\partial \dot{k}_\sigma(k)}{\partial \sigma}. \tag{27}$$

Suppose $s^w \neq s^r$ and consider a first-order Taylor approximation of $\partial \dot{k}_\sigma(k)/\partial\sigma$ about $\bar{k}$. As to the wage, we obtain the approximation

$$\frac{\partial w_\sigma\left(k\right)}{\partial \sigma} = \frac{\partial w_\sigma\left(\bar{k}\right)}{\partial \sigma} + \frac{\partial w_\sigma'\left(\bar{k}\right)}{\partial \sigma}(k - \bar{k}) = 0 - \bar{k}\,\frac{\partial f_\sigma''(\bar{k})}{\partial \sigma}(k - \bar{k}). \tag{28}$$

Here, we use (9) and $w'_\sigma(k) = -kf''_\sigma(\bar{k})$.

As to the real rate of return on capital, the same approximation gives

$$\frac{\partial r_\sigma(k)}{\partial \sigma} = \frac{\partial r_\sigma\left(\bar{k}\right)}{\partial \sigma} + \frac{\partial r'_\sigma\left(\bar{k}\right)}{\partial \sigma}(k-\bar{k}) = 0 + \frac{\partial f''_\sigma(\bar{k})}{\partial \sigma}(k-\bar{k}). \tag{29}$$

The derivation uses from (10), $r'_\sigma(k) = f''_\sigma(\bar{k})$.

Upon substitution of (28) and (29) in (27) gives the desired Taylor-approximation as

$$\begin{aligned}\frac{\partial \dot{k}_\sigma(k)}{\partial \sigma} &= s^w\left(-\bar{k}\right)\frac{\partial f''_\sigma(\bar{k})}{\partial \sigma}(k-\bar{k}) + s^r\bar{k}\,\frac{\partial f''_\sigma(\bar{k})}{\partial \sigma}(k-\bar{k}),\\ &= \bar{k}\,\frac{\partial f''_\sigma(\bar{k})}{\partial \sigma}\left(s^r - s^w\right)(k-\bar{k}).\end{aligned} \tag{30}$$

If $s^w = s^r = s$, then the first-order approximation of (14) vanishes. However, a second-order Taylor approximation delivers the result. To see this, observe that (12) becomes

$$\frac{\partial \dot{k}}{\partial \sigma} = \frac{\partial \dot{k}_\sigma(k)}{\partial \sigma} = s\,\frac{\partial f_\sigma(k)}{\partial \sigma}. \tag{31}$$

Consider the approximation

$$\frac{\partial \dot{k}_\sigma(k)}{\partial \sigma} \simeq s\,\frac{\partial f_\sigma\left(\bar{k}\right)}{\partial \sigma} + s\,\frac{\partial f'_\sigma\left(\bar{k}\right)}{\partial \sigma}(k-\bar{k}) + \frac{s}{2}\,\frac{\partial f''_\sigma\left(\bar{k}\right)}{\partial \sigma}(k-\bar{k})^2. \tag{32}$$

Due to the normalization of the CES, the first two terms vanish and (15) obtains.

QED.

5.3 Proof of Proposition 2

To study the sign of (17) expand $\partial \dot{y}_\sigma(k)/\partial \sigma$ about $\bar{k}$. Due to our normalization $\partial \dot{y}_\sigma\left(\bar{k}\right)/\partial \sigma = 0$, hence

$$\frac{\partial \dot{y}_\sigma(k)}{\partial \sigma} \simeq \frac{\partial \dot{y}'_\sigma\left(\bar{k}\right)}{\partial \sigma}\left(k-\bar{k}\right). \tag{33}$$

Since $\dot{y}'_\sigma(k) = f''_\sigma(k)\,\dot{k}_\sigma(k) + f'_\sigma(k)\,\partial \dot{k}_\sigma(k)/\partial k$, we have

$$\frac{\partial \dot{y}'_\sigma\left(\bar{k}\right)}{\partial \sigma} = \left.\left(\frac{\partial f''_\sigma}{\partial \sigma}\dot{k}_\sigma + \frac{\partial f'_\sigma}{\partial \sigma}\frac{\partial \dot{k}_\sigma}{\partial k} + f''_\sigma\frac{\partial \dot{k}_\sigma}{\partial \sigma} + f'_\sigma\frac{\partial \dot{k}'_\sigma}{\partial \sigma}\right)\right|_{k=\bar{k}}. \tag{34}$$

One readily verifies that the second and the third term in (34) are nil. Then, we obtain

$$\begin{aligned}\frac{\partial \dot{y}'_\sigma\left(\bar{k}\right)}{\partial\sigma} &= \frac{\partial f''_\sigma(\bar{k})}{\partial\sigma}\left(s^w w_\sigma(\bar{k}) + s^r\bar{k}\,r_\sigma(\bar{k}) - nk\right) + f'_\sigma\left[s^w\frac{\partial w_\sigma(\bar{k})}{\partial\sigma} + s^r\bar{k}\,\frac{\partial r_\sigma(\bar{k})}{\partial\sigma}\right],\\ &= \frac{\partial f''_\sigma(\bar{k})}{\partial\sigma}\left[s^w\bar{y}\,(1-2\bar{\pi}) + s^r\bar{y}\,2\bar{\pi} - n\bar{k}\right],\\ &= \frac{\partial f''_\sigma(\bar{k})}{\partial\sigma}\left[\dot{k}_\sigma(\bar{k}) + \bar{y}\,\bar{\pi}\,(s^r - s^w)\right],\end{aligned}$$

where we use (9), (10), and the definition of $\dot{k}(\bar{k})$. From Lemma 1 we know that $\partial f''_\sigma(\bar{k})/\partial\sigma > 0$. Hence, we find

$$\frac{\partial \dot{y}'_\sigma\left(\bar{k}\right)}{\partial\sigma} \gtreqless 0 \quad\Leftrightarrow\quad \left(\dot{k}_\sigma(\bar{k}) + \bar{y}\,\bar{\pi}\,(s^r - s^w)\right) \gtreqless 0. \tag{35}$$

Replacing $\dot{k}_\sigma(\bar{k})$, we find for the right-hand side of (33)

$$\frac{\partial \dot{y}'_\sigma\left(\bar{k}\right)}{\partial\sigma}\left(k-\bar{k}\right) \gtreqless 0 \quad\Leftrightarrow\quad \left(2\bar{\pi}\left(\frac{s^r}{s^w}-1\right) + \frac{1}{s^w\bar{y}}\left(s^w\bar{y} - n\bar{k}\right)\right)\left(k-\bar{k}\right) \gtreqless 0. \tag{36}$$

Hence, (18) holds.

QED.

5.4 Proof of Proposition 3

Consider the case where $s^r > s^w$. We show that $\partial\dot{k}(k)/\partial\sigma > 0$ for all $k > \bar{k}$. In view of (12), this requires

$$s^r\left[\pi_\sigma\frac{\partial f_\sigma(k)}{\partial\sigma} + f_\sigma(k)\,\frac{\partial\pi_\sigma}{\partial\sigma}\right] > s^w\left[\frac{\partial f_\sigma(k)}{\partial\sigma}(\pi_\sigma - 1) + f_\sigma(k)\,\frac{\partial\pi_\sigma}{\partial\sigma}\right] \tag{37}$$

or

$$s^r > s_c \equiv \max\left[s^w\left(1 - \frac{\frac{\partial f_\sigma(k)}{\partial\sigma}}{\pi_\sigma\frac{\partial f_\sigma(k)}{\partial\sigma} + f_\sigma(k)\,\frac{\partial\pi_\sigma}{\partial\sigma}}\right), 0\right]. \tag{38}$$

According to (8) the efficiency effect is strictly positive for $k \neq \bar{k}$. Moreover, in accordance with (13) the distribution effect is positive whenever $k > \bar{k}$. Therefore, the term in parentheses on the right-hand side of (38) is strictly smaller than one for all $k > \bar{k}$. Hence, $s_c < s^w$ such that $s^r > s^w$ is sufficient for $\partial\dot{k}(k)/\partial\sigma > 0$ to hold for all $k > \bar{k}$.

Turning to the evolution of per-capita income, we first state and prove the following claim.

Claim 1 Let

$$\Phi := \pi_\sigma \ln\left(\frac{\overline{\pi}}{\pi_\sigma}\right) + (1-\pi_\sigma)\ln\left(\frac{1-\overline{\pi}}{1-\pi_\sigma}\right) \tag{39}$$

where $\overline{\pi} = \pi_\sigma\ \left(\overline{k}\right)$. It holds that

$$\frac{\partial f'_\sigma(k)}{\partial\sigma} = \frac{f'_\sigma}{\sigma^2}\frac{1}{\psi^2}\left[\psi^2(1-\pi_\sigma)\ln\left(\frac{k}{\overline{k}}\right) - \Phi\right] \tag{40}$$

and

$$\frac{\partial f'_\sigma(k)}{\partial\sigma} > 0 \text{ for } k > \overline{k}. \tag{41}$$

Proof of Claim 1

From $\pi_\sigma \equiv k f'_\sigma(k)/f_\sigma(k)$ it follows that

$$\frac{\partial\pi_\sigma}{\partial\sigma} = \frac{k}{(f_\sigma)^2}\left(\frac{\partial f'_\sigma}{\partial\sigma}f_\sigma - \frac{\partial f_\sigma}{\partial\sigma}f'_\sigma\right)$$

or

$$\frac{\partial f'_\sigma(k)}{\partial\sigma} = \frac{\partial\pi_\sigma}{\partial\sigma}\frac{f_\sigma}{k} + \frac{\partial f_\sigma}{\partial\sigma}\frac{f'_\sigma}{f_\sigma}. \tag{42}$$

Next, we make use of (13) and of equation 11 in Klump and de La Grandville (2000) stating that

$$\frac{\partial f_\sigma(k)}{\partial\sigma} = -\frac{f_\sigma}{\sigma^2}\frac{1}{\psi^2}\Phi. \tag{43}$$

Plugging (13) and (43) into (42) gives after some simple algebraic manipulation (40).

Equation (41) follows from the facts that for $k > \overline{k}$, $\ln\left(k/\overline{k}\right) > 0$ and $\Phi < 0$ (see, equation 13 in Klump and de La Grandville (2000)).

QED.

To show that $\partial\dot{y}(k)/\partial\sigma > 0$ for all $k > \bar{k}$ consider the terms on the right-hand side of (17). In view of Claim 2 and the fact that the economy grows, the first term is strictly positive for $k > \bar{k}$. As shown above, the same is true for $\partial\dot{k}(k)/\partial\sigma$.

The results for $s^r = s^w$ follow immediately from Klump and de La Grandville (2000), Theorem 1.

QED.

5.5 Proof of Proposition 4

Since $\dot{k}_{\sigma_1}\left(\bar{k}\right) = \dot{k}_{\sigma_2}\left(\bar{k}\right) > 0$ it follows that $k^* > \bar{k}$. Therefore, all derivatives with respect to σ that appear on the right-hand side of (22) are strictly positive. Since $f_\sigma'' < 0$, we have $dk^*/d\sigma > 0$. As to the steady-state per-capita income, we have $y^* = f_\sigma(k^*)$ such that $\partial y^*/\partial\sigma = \partial f_\sigma/\partial\sigma + f_\sigma'\, dk^*/d\sigma > 0$.

QED.

References

Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow (1961): "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, 43(3), 225–250.

Bernanke, B. S., and R. S. Gürkaynak (2001): "Is Growth Exogenous? Taking Mankiw, Romer, and Weil Seriously," in *NBER Macroeconomics Annual 2001*, ed. by B. S. Bernanke, and K. S. Rogoff. MIT Press, Cambridge, MA.

de La Grandville, O. (1989): "In Quest of the Slutsky Diamond," *American Economic Review*, 79(3), 468–481.

Diamond, P. A. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55(5), 1126–1150.

Irmen, A. (2003): "On the Growth Effects of the Elasticity of Substitution," *mimeo, University of Mannheim.*

Kaldor, N. (1956): "Alternative Theories of Distribution," *Review of Economic Studies*, 23(2), 83–100.

Klump, R., and O. de La Grandville (2000): "Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions," *American Economic Review*, 90(1), 282–291.

Klump, R., and M. Saam (2008): "Calibration of Normalized CES Production Functions in Dynamic Models," *Economics Letters, forthcoming.*

Miyagiwa, K., and C. Papageorgiou (2003): "Elasticity of Substitution and Growth: Normalized CES in the Diamond Model," *Economic Theory*, 21(1), 155–165.

Solow, R. M. (1956): "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, 70(1), 65–94.

Uzawa, H. (1961): "On a Two-Sector Model of Economic Growth I," *Review of Economic Studies*, 29(1), 40–47.